

# Polynomiality of the $q, t$ -Kostka Revisited

by

**Adriano M. Garsia** <sup>†</sup>

*University of California San Diego*

and

**Mike Zabrocki**

*Université du Québec à Montréal*

## Abstract.

Let  $K(q, t) = \|K_{\lambda\mu}(q, t)\|_{\lambda, \mu}$  be the Macdonald  $q, t$ -Kostka matrix and  $K(t) = K(0, t)$  be the matrix of the Kostka-Foulkes polynomials  $K_{\lambda\mu}(t)$ . In this paper we present a new proof of the polynomiality of the  $q, t$ -Kostka coefficients that is both short and elementary. More precisely, we derive that  $K(q, t)$  has entries in  $\mathbb{Z}[q, t]$  directly from the fact that the matrix  $K(t)^{-1}$  has entries in  $\mathbb{Z}[t]$ . The proof uses only identities that can be found in the original paper [7] of Macdonald.

## Introduction

The polynomiality problem for the  $q, t$ -Kostka coefficients [11], was posed by Macdonald in the fall 1988 meeting of the Lotharingian seminar. It remained open for quite a few years, when suddenly in 1996, several proofs of varied difficulty appeared in a period of only a few months. At the present there are three basically different approaches to proving the polynomiality of the  $q, t$ -Kostka coefficients:

- (1) Via plethystic formulas (Garsia-Tesler [4], Garsia-Remmel [3]).
- (2) Via vanishing properties (Sahi [13] and Knop [7],[8]).
- (3) Via Rodriguez formulas (Lapointe-Vinet [10], Kirillov-Noumi [6])

Each of these approaches has its own special advantages. The plethystic approach led to very efficient algorithms for computing these coefficients and ultimately produced some remarkably simple explicit formulas [2]. The vanishing properties approach led to the discovery some basic non-symmetric variants of the Macdonald polynomials with remarkable combinatorial implications that still remain to be fully explored. The approach via Rodriguez formulas stems from a pioneering paper of Lapointe-Vinet [9] on Jack-Polynomials. Although originally it was based on deep affine Hecke algebra identities, eventually the idea led to some of the most elementary proofs of the polynomiality result (see [10] and [6]). In particular it produced a family of symmetric function operators  $\{B_k^{q,t}\}_{k=1,2,\dots}$ , which permitted the construction of the Macdonald “integral forms”  $J_\mu(x; q, t)$ , one part at the time, starting from 1, according to an identity of the form

$$J_{\mu'}(x; q, t) = B_{\mu_1}^{q,t} B_{\mu_2}^{q,t} \cdots B_{\mu_k}^{q,t} \mathbf{1} .$$

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Our main contribution here is a remarkably simple argument which shows that families of operators  $B_k^{q,t}$  yielding such a formula, may obtained by  $q$ -twisting in a minor way **any** sequence of operators  $\{B_k^t\}_{k=1,2,\dots}$  which yields the analogous formula

$$Q_{\mu'}(x; t) = B_{\mu_1}^t B_{\mu_2}^t \cdots B_{\mu_k}^t \mathbf{1} .$$

for the Hall-Littlewood polynomial  $Q_{\mu}(x; q, t)$ . As a byproduct we obtain that the polynomiality of the  $q, t$ -Kostka is an immediate consequence of the polynomiality of the Kostka-Foulkes coefficients. What is surprising is that this fact was missed for so many years by researchers in this area. What might be even more surprising is that we obtain a remarkably general result by further simplifying some of the arguments used in [10] and [6]. To give a more precise description of our results we need some notation.

We shall deal with identities in the algebra  $\Lambda$  of symmetric functions in a finite or infinite alphabet  $X = \{x_1, x_2, x_3, \dots\}$  with coefficients in the field of rational functions  $\mathbb{Q}(q, t)$ . We also denote by  $\Lambda_{\mathbb{Z}[q,t]}$  the algebra of symmetric functions in  $X$  with coefficients in  $\mathbb{Z}[q, t]$

An essential notational tool in our presentation is the notion of “*plethystic substitution*” and we need to recall its definition. Briefly, if  $E = E(t_1, t_2, t_3, \dots)$  is a given formal series in the variables  $t_1, t_2, t_3, \dots$  (which may include the parameters  $q, t$ ) and  $f \in \Lambda$  has been expressed in terms of the power basis in the form

$$F = Q(p_1, p_2, p_3, \dots)$$

then the “*plethystic substitution*” of  $E$  in  $F$ , denoted  $F[E]$ , is simply defined by setting

$$F[E] = Q(p_1, p_2, p_3, \dots) \Big|_{p_k \rightarrow E(t_1^k, t_2^k, t_3^k, \dots)} . \quad \text{I.1}$$

This operation is easily programmed in any symbolic manipulation software which includes a symmetric function package. We shall adopt the convention that inside the plethystic brackets  $[ ]$ ,  $X$  and  $X_n$  respectively stand for  $x_1 + x_2 + x_3 + \dots$  and  $x_1 + x_2 + \dots + x_n$ . A similar notation is adopted when we work with any other alphabet. Note that if  $P \in \Lambda$  then  $P[X_n]$  simply means  $P(x_1, x_2, \dots, x_n)$ . Plethystic substitution need not be restricted to symmetric polynomials. In fact, the substitution in I.1 makes sense even when  $Q$  is a formal power series. In this vein, we shall systematically use the symbol  $\Omega[X]$  to represent the symmetric function series

$$\Omega[X] = \prod_i \frac{1}{1 - x_i} = \exp \left( \sum_{k \geq 1} \frac{p_k[X]}{k} \right) . \quad \text{I.2}$$

This notation is particularly convenient since for any two alphabets  $X, Y$  from I.1 and I.2 we easily derive that

$$\Omega[X + Y] = \Omega[X] \Omega[Y] , \quad \Omega[X - Y] = \Omega[X] / \Omega[Y] \quad \text{I.3}$$

In particular we see that the Macdonald kernel  $\Omega_{q,t}(x_1, x_2, \dots, x_n; y_1, y_2, \dots, y_k)$  (see [12] (2.5) p. 309) may be expressed in the compact form.

$$\Omega_{q,t}(x_1, x_2, \dots, x_n; y_1, y_2, \dots, y_k) = \Omega[X_n Y_k \frac{1-t}{1-q}] \quad \text{I.4}$$

We represent partitions here by their French Ferrers diagram, that is with rows decreasing from bottom to top. To express that a certain lattice cell  $s$  belongs to the Ferrers diagram of a partition  $\lambda$  we shall simply write  $s \in \lambda$ . More generally we shall always identify partitions with their corresponding diagrams. The number of rows of the Ferrers diagram of a partition  $\mu$  will be called the *length* of  $\mu$  and denoted  $l(\mu)$ . We shall often make use of the operation of prepending a column of length  $k$  to the diagram of a partition  $\mu$  of length  $\leq k$ , for convenience we shall denote the resulting partition by the symbol  $\mu + 1^k$ .

Let us recall that the *integral forms*  $J_\lambda(x; q, t)$  of the Macdonald polynomials  $P_\lambda(x; q, t)$  and  $Q_\lambda(x; q, t)$  are defined (see [12] p. 352) by setting

$$J_\lambda(x; q, t) = h_\lambda(q, t) P_\lambda(x; q, t) = h'_\lambda(q, t) Q_\lambda(x; q, t) \quad \text{I.5}$$

with

$$h_\lambda(q, t) = \prod_{s \in \lambda} (1 - q^{a_\lambda(s)} t^{l_\mu(s)+1}), \quad h'_\lambda(q, t) = \prod_{s \in \lambda} (1 - q^{a_\lambda(s)+1} t^{l_\mu(s)}) \quad \text{I.6}$$

where, for a cell  $s \in \lambda$ ,  $a_\lambda(s)$  and  $l_\lambda(s)$  represent the *arm* and *leg* of  $s$  in  $\lambda$ , that is the number of cells of  $\lambda$  that are respectively strictly *EAST* and *NORTH* of  $s$ . Recall that Garsia-Haiman [1] introduce the modified versions  $H_\mu[X; q, t]$  of the integral forms by setting

$$H_\mu[X; q, t] = J_\mu[\frac{X}{1-t}; q, t] \quad \text{I.7}$$

These polynomials offer direct access to the Macdonald  $q, t$ -Kostka coefficients because their Schur function expansion reduces to

$$H_\mu[X; q, t] = \sum_{\lambda} S_\lambda[X] K_{\lambda\mu}(q, t)$$

This follows immediately from the definition of  $K_{\lambda\mu}(q, t)$ , given in [12] ((8.11) p. 354), as the coefficients appearing in the expansion

$$J_\mu[X; q, t] = \sum_{\lambda} S_\lambda[X(1-t)] K_{\lambda\mu}(q, t) \quad \text{I.8}$$

It is also shown in [12] ((8.12) p. 354) that we have

$$J_\mu[X; 0, t] = Q_\mu[X; t] = \sum_{\lambda} S_\lambda[X(1-t)] K_{\lambda\mu}(t) \quad \text{I.9}$$

where  $Q_\lambda[X; t]$  is the Hall-Littlewood polynomial and  $K_{\lambda\mu}(t)$  denotes the corresponding *Kostka-Foulkes* coefficient. We shall also set

$$H_\mu[X; t] = Q_\lambda\left[\frac{X}{1-t}; t\right] = \sum_{\lambda} S_\lambda[X] K_{\lambda\mu}(t) . \quad \text{I.10}$$

It develops that the polynomiality of the  $K_{\lambda\mu}(q, t)$  is an immediate consequence of the following general result first proved in [14].

**Theorem I.1**

*For any linear operator  $V$  acting on  $\Lambda$  and  $P \in \Lambda$  set*

$$\widetilde{V}^q P[X] = V^Y P[qX + (1-q)Y] \Big|_{Y=X} \quad \text{I.11}$$

*where  $V^Y$  is simply  $V$  acting on polynomials in the  $Y$  variables. This given, if  $G_k = G_k(X, t)$  is any linear operator on  $\Lambda$  with the property that*

$$G_k H_\mu[X; t] = H_{\mu+1^k}[X; t] \quad \text{for all } \mu \text{ of length } \leq k \quad \text{I.12}$$

*then  $\widetilde{G}_k^q$  has the property*

$$\widetilde{G}_k^q H_\mu[X; q, t] = H_{\mu+1^k}[X; q, t] \quad \text{for all } \mu \text{ of length } \leq k . \quad \text{I.13}$$

*In particular, the modified Macdonald polynomial  $H_\mu[X; q, t]$  may be obtained from the “Rodriguez” formula:*

$$H_\mu[X; q, t] = \widetilde{G}_{\mu'_1}^q \widetilde{G}_{\mu'_2}^q \cdots \widetilde{G}_{\mu'_h}^q \mathbf{1} \quad \text{I.14}$$

*where  $\mu' = (\mu'_1, \mu'_2, \dots, \mu'_h)$  denotes the conjugate of  $\mu$ .*

Our main contribution here is a simple, direct and elementary proof of this result which only uses identities given in the original paper of Macdonald.

Note that since the Kostka-Foulkes matrix  $K(t) = \|K_{\lambda\mu}(t)\|_{\lambda\mu}$  is unitriangular it follows that its inverse  $H(t) = K(t)^{-1}$  has entries in  $\mathbb{Z}[t]$ . This implies that the “trivial” operator  $TG_k = TG_k(X; t)$  defined by setting for the  $\{H_\mu[X; t]\}_\mu$  basis

$$TG_k H_\mu[X; t] = \begin{cases} H_{\mu+1^k}[X; t] & \text{if } l(\mu) \leq k \\ 0 & \text{otherwise} \end{cases} \quad \text{I.15}$$

acts integrally on the Schur basis. This given, we see that the desired result

$$K_{\lambda\mu}(q, t) \in \mathbb{Z}[q, t] \quad \text{I.16}$$

is an immediate consequence of I.14 with  $G = TG$ .

This paper is divided in two sections. In the first section we prove Theorem I.1, and in the second section we give a number of applications, including the explicit derivation of the action of a variant of the operator  $TG$  on the monomial basis.

## 1. Rodriguez operators for the Integral Forms

We shall start by proving a result analogous to Theorem I.1 for the Macdonald integral forms. To this end we shall need a number of auxiliary results.

### Proposition 1.1

*If  $V(X; t)$  is a linear operator on symmetric functions in  $X$  that depends only on  $t$  and we set*

$$\tilde{V}^q(X; t)P[X] = V(Z; t)^Z P[qX + (1 - q)Z] \Big|_{Z=X} \quad 1.1$$

*then*

$$\frac{\tilde{V}^q(X; t)\Omega\left[XY\frac{1-t}{1-q}\right]}{\Omega\left[XY\frac{1-t}{1-q}\right]} = \frac{V(X; t)^X\Omega[XY(1-t)]}{\Omega[XY(1-t)]} \quad 1.2$$

*in particular this ratio is independent of  $q$ .*

### Proof

We have

$$\begin{aligned} \tilde{V}^q(X; t)\Omega\left[XY\frac{1-t}{1-q}\right] &= V(Z; t)^Z\Omega\left[(qX + (1 - q)Z)Y\frac{1-t}{1-q}\right] \Big|_{Z=X} \\ &= \Omega\left[qXY\frac{1-t}{1-q}\right] V(X; t)^X\Omega\left[(1 - q)XY\frac{1-t}{1-q}\right] \\ &= \Omega\left[XY\frac{1-t}{1-q}\right]\Omega\left[(q - 1)XY\frac{1-t}{1-q}\right] V(X; t)^X\Omega[XY(1 - t)] \\ &= \Omega\left[XY\frac{1-t}{1-q}\right]\Omega[XY(t - 1)] V(X; t)^X\Omega[XY(1 - t)] \end{aligned}$$

This proves 1.2.

Recall that for an alphabet  $Y_k = y_1 + y_2 + \dots + y_k$  and an interval  $I \subseteq [1, k] = \{1, 2, \dots, k\}$  Macdonald sets (see (3.5) [12] p. 315).

$$A_I[Y_k; t] = t^{\binom{|I|}{2}} \prod_{\substack{i \in I \\ j \in [1, k] - I}} \frac{ty_i - y_j}{y_i - y_j}$$

Note also that for any symmetric polynomial  $P[Y_k]$ , the operator  $T_I^q$  which replaces  $y_i$  by  $qy_i$  for every  $i \in I$  may be written in the form

$$T_I^q P[Y_k] = P[Y_k + (q - 1)Y_I] \quad 1.3$$

where

$$Y_I = \sum_{i \in I} y_i .$$

This given, we have the following identity due to Kirillov-Noumi [6].

**Proposition 1.2**

Let  $M^{(Y_k)}(u)$  denote the Macdonald operator acting on polynomials in the alphabet  $Y_k$ . That is

$$M^{(Y_k)}(u) = \sum_{r=0}^k u^r \sum_{\substack{I \subseteq [1, k] \\ |I|=r}} A_I[Y_k; t] T_I^q$$

then for  $k \leq n$

$$M^{(Y_k)}(-1) \Omega \left[ X_n Y_k \frac{1-t}{1-q} \right] = y_1 y_2 \cdots y_k \sum_{l(\mu) \leq k} J_{\mu+1^k}[X_n; q, t] \frac{P_\mu[Y_k; q, t]}{h'_\mu(q, t)} \quad 1.4$$

**Proof**

We start with the Macdonald *Cauchy*-identity from equation (4.13) in [12]

$$\Omega \left[ X_n Y_k \frac{1-t}{1-q} \right] = \sum_{l(\lambda) \leq k} Q_\lambda[X_n; q, t] P_\lambda[Y_k; q, t]$$

the summation being only over partitions  $\lambda$  of length  $\leq k$  because  $P_\lambda$  vanishes when evaluated on an alphabet whose cardinality is smaller than the length of  $\lambda$ . Applying  $M^{(Y_k)}(u)$  to both sides of this identity and using Theorem (4.15) of [12] (p. 324) we get that

$$M^{(Y_k)}(u) \Omega \left[ X_n Y_k \frac{1-t}{1-q} \right] = \sum_{l(\lambda) \leq k} Q_\lambda[X_n; q, t] \left( \prod_{i=1}^k (1 + u t^{k-i} q^{\lambda_i}) \right) P_\lambda[Y_k; q, t],$$

Now note that if  $l(\lambda) < k$  then the term corresponding to  $\lambda$  in this sum will vanish if we set  $u = -1$ . We thus obtain

$$M^{(Y_k)}(-1) \Omega \left[ X_n Y_k \frac{1-t}{1-q} \right] = \sum_{l(\lambda)=k} Q_\lambda[X_n; q, t] \left( \prod_{i=1}^k (1 - t^{k-i} q^{\lambda_i}) \right) P_\lambda[Y_k; q, t] \quad 1.5$$

Now it follows from theorem (4.17) of [12] (p. 325) that if  $l(\lambda) = k$  then

$$P_\lambda[Y_k; q, t] = y_1 y_2 \cdots y_k P_\mu[Y_k; q, t]$$

with

$$\mu = (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_k - 1) .$$

Thus we may rewrite 1.5 in the form

$$M^{(Y_k)}(-1)\Omega[X_n Y_k \frac{1-t}{1-q}] = y_1 y_2 \cdots y_k \sum_{l(\mu) \leq k} Q_{\mu+1^k}[X_n; q, t] \left( \prod_{i=1}^k (1 - t^{k-i} q^{\mu_i+1}) \right) P_\mu[Y_k; q, t] \quad 1.6$$

Now from I.5 and I.6 we derive that

$$Q_{\mu+1^k}[X_n; q, t] \left( \prod_{i=1}^k (1 - t^{k-i} q^{\mu_i+1}) \right) = \frac{J_{\mu+1^k}[X_n; q, t]}{h'_\mu(q, t)}.$$

Thus the desired identity in 1.4 immediately follows upon using this in 1.6.

**Proposition 1.3**

$$\frac{M^{(Y_k)}(u)\Omega[X_n Y_k \frac{1-t}{1-q}]}{\Omega[X_n Y_k \frac{1-t}{1-q}]} = \sum_{r=0}^k u^r \sum_{\substack{I \subseteq [1, k] \\ |I|=r}} A_I[Y_k; t] \Omega[X_n Y_I(t-1)], \quad 1.7$$

in particular this ratio is independent of  $q$ .

**Proof**

$$\begin{aligned} M^{(Y_k)}(u)\Omega[X_n Y_k \frac{1-t}{1-q}] &= \sum_{r=0}^k u^r \sum_{\substack{I \subseteq [1, k] \\ |I|=r}} A_I[Y_k; t] \Omega[X_n (Y_k + (q-1)Y_I) \frac{1-t}{1-q}] \\ &= \Omega[X_n Y_k \frac{1-t}{1-q}] \sum_{r=0}^k u^r \sum_{\substack{I \subseteq [1, k] \\ |I|=r}} A_I[Y_k; t] \Omega[X_n Y_I(t-1)] \end{aligned}$$

and this proves 1.7.

**Theorem 1.1**

If  $B_k(X; t)$  is any operator with the property that

$$B_k(X; t)J_\mu[X; 0, t] = J_{\mu+1^k}[X; 0, t] \quad \text{for all } l(\mu) \leq k \quad 1.8$$

then

$$\widetilde{B}_k^q(X; t)J_\mu[X; q, t] = J_{\mu+1^k}[X; q, t] \quad \text{for all } l(\mu) \leq k \quad 1.9$$

**Proof**

Suppose we show that

$$y_1 y_2 \cdots y_k \widetilde{B}_k^q(X_n; t)^{X_n} \Omega[X_n Y_k \frac{1-t}{1-q}] = M^{(Y_k)}(-1)\Omega[X_n Y_k \frac{1-t}{1-q}], \quad 1.10$$

then by combining this with Proposition 1.2 we get that

$$\begin{aligned} y_1 y_2 \cdots y_k \sum_{l(\mu) \leq k} \widetilde{B}_k^q(X_n; t)^{X_n} J_\mu[X_n; q, t] \frac{P_\mu[Y_k; q, t]}{h'_\mu(q, t)} \\ = y_1 y_2 \cdots y_k \sum_{l(\mu) \leq k} J_{\mu+1^k}[X_n; q, t] \frac{P_\mu[Y_k; q, t]}{h'_\mu(q, t)} \end{aligned}$$

and 1.9 then follows by equating coefficients of  $P_\mu[Y_k; q, t]$ . To show 1.10 we need only verify that

$$y_1 y_2 \cdots y_k \frac{\widetilde{B}_k^q(X_n; t)^{X_n} \Omega \left[ X_n Y_k \frac{1-t}{1-q} \right]}{\Omega \left[ X_n Y_k \frac{1-t}{1-q} \right]} = \frac{M^{(Y_k)}(-1) \Omega \left[ X_n Y_k \frac{1-t}{1-q} \right]}{\Omega \left[ X_n Y_k \frac{1-t}{1-q} \right]}, \quad 1.11$$

and since we have shown that both sides of this equation are independent of  $q$ , we need only verify this equality at  $q = 0$ . However, the hypothesis in 1.8 yields that

$$\begin{aligned} y_1 y_2 \cdots y_k \frac{\widetilde{B}_k^q(X_n; t)^{X_n} \Omega \left[ X_n Y_k \frac{1-t}{1-q} \right]}{\Omega \left[ X_n Y_k \frac{1-t}{1-q} \right]} \Big|_{q=0} \\ = \frac{y_1 y_2 \cdots y_k}{\Omega[X_n Y_k(1-t)]} \sum_{l(\mu) \leq k} B_k(X_n; t)^{X_n} J_\mu[X_n; 0, t] \frac{P_\mu[Y_k; 0, t]}{h'_\mu(0, t)} \\ = \frac{y_1 y_2 \cdots y_k}{\Omega[X_n Y_k(1-t)]} \sum_{l(\mu) \leq k} J_{\mu+1^k}[X_n; 0, t] \frac{P_\mu[Y_k; 0, t]}{h'_\mu(0, t)} \end{aligned}$$

and again by Proposition 1.3 we see that this is precisely

$$\frac{M^{(Y_k)}(-1) \Omega \left[ X_n Y_k \frac{1-t}{1-q} \right]}{\Omega \left[ X_n Y_k \frac{1-t}{1-q} \right]} \Big|_{q=0}.$$

This completes the proof of 1.9.

Before we can proceed with the proof of Theorem I.1 we need one more auxiliary result. To begin with it will be convenient to consider the substitution  $X \rightarrow X/(1-t)$  as a linear operator on symmetric functions. More precisely, for any symmetric polynomial  $P$  and any alphabet  $X$  we set

$$F^t P[X] = P[X(1-t)]$$

Now a somewhat surprising development is that the operation  $V \rightarrow \widetilde{V}^q$  defined in I.11 commutes with conjugation by  $F^t$ . In fact, we may state

**Proposition 1.4**

*For any linear operator  $V$  acting on  $\Lambda$  and any polynomial  $P \in \Lambda$  we have*

$$F^t \widetilde{V}^q F^{t-1} P[X] = (F^t \widetilde{V} F^{t-1})^q P[X] \quad 1.12$$

**Proof**



It is sufficient to prove 1.12 for the Schur basis. Note that for any partition  $\lambda$ , the addition formula for Schur functions gives

$$\begin{aligned} (F^t \widetilde{V F^{t-1}})^q S_\lambda[X] &= (F^t V F^{t-1})^Y S_\lambda[qX + (1-q)Y] \Big|_{Y=X} \\ &= \sum_{\mu \subseteq \lambda} S_{\lambda/\mu}[qX] (F^t V F^{t-1})^Y S_\mu[(1-q)Y] \Big|_{Y=X} \\ &= \sum_{\mu \subseteq \lambda} S_{\lambda/\mu}[qX] (F^t V F^{t-1})^X S_\mu[(1-q)X] . \end{aligned} \tag{1.13}$$

In the same vein we see that the left hand side of 1.12, for  $P = S_\lambda$ , gives

$$\begin{aligned} F^t \widetilde{V^q F^{t-1}} S_\lambda[X] &= F^t \widetilde{V^q} S_\lambda[\frac{X}{1-t}] \\ &= F^t V^Y S_\lambda \left[ \frac{qX + (1-q)Y}{1-t} \right] \Big|_{Y=X} \\ &= F^t \sum_{\mu \subseteq \lambda} S_{\lambda/\mu} \left[ \frac{qX}{1-t} \right] V^Y S_\mu \left[ \frac{(1-q)Y}{1-t} \right] \Big|_{Y=X} \\ &= \sum_{\mu \subseteq \lambda} S_{\lambda/\mu}[qX] F^t V^X S_\mu \left[ \frac{(1-q)X}{1-t} \right] \end{aligned}$$

and it is easily seen that this is another way to write the last expression in 1.13.

We are now in a position to give our

### Proof of Theorem I.1

By assumption we have

$$G_k H_\mu[X; t] = H_{\mu+1^k}[X; t] \quad \text{for all } \mu \text{ of length } \leq k \tag{1.14}$$

Thus from I.10 we derive that

$$G_k F^{t-1} Q_\mu[X; t] = F^{t-1} Q_{\mu+1^k}[X; t] .$$

Now this, using I.9, may be rewritten as

$$F^t G_k F^{t-1} J_\mu[X; 0, t] = J_{\mu+1^k}[X; 0, t] \quad \text{for all } \mu \text{ of length } \leq k .$$

In other words the operator

$$B_k = B_k(X; t) = F^t G_k F^{t-1}$$

satisfies the hypothesis of Theorem 1.1. It then follows that

$$(F^t \widetilde{G_k F^{t-1}})^q J_\mu[X; q, t] = J_{\mu+1^k}[X; q, t] \quad \text{for all } \mu \text{ of length } \leq k$$

But Proposition 1.4 yields

$$F^t \widetilde{G_k}^q F^{t-1} J_\mu[X; q, t] = J_{\mu+1^k}[X; q, t] \quad \text{for all } \mu \text{ of length } \leq k$$

and I.7 shows that this is just another way of writing I.13, completing the proof of Theorem I.1.

**Remark 1.1**

We should note that any symmetric function operator  $V(q, t) = V(x; q, t)$  of the form

$$V(q, t) = \sum_{I \subseteq [1, k]} a_I(x, t) T_I^q . \quad 1.15$$

(in particular the Macdonald operator) satisfies the identity

$$V(q, t) = \widetilde{V(0, t)}^q \quad 1.16$$

In fact, since for any  $P \in \Lambda$  we have

$$V(q, t)P[X] = \sum_{I \subseteq [1, k]} a_I(x, t) P[X + (q - 1)X_I] .$$

then

$$V(0, t)P[X] = \sum_{I \subseteq [1, k]} a_I(x, t) P[X - X_I] .$$

Thus

$$\begin{aligned} \widetilde{V(0, t)}^q P[X] &= V(0, t)^Y P[qX + (1 - q)Y] \Big|_{Y=X} \\ &= \sum_{I \subseteq [1, k]} a_I(y, t) P[qX + (1 - q)(Y - Y_I)] \Big|_{Y=X} \\ &= \sum_{I \subseteq [1, k]} a_I(x, t) P[qX + (1 - q)(X - X_I)] \\ &= \sum_{I \subseteq [1, k]} a_I(x, t) P[X + (q - 1)X_I] \end{aligned}$$

which is 1.16.

We then see that Propositions 1.1 and 1.3 are both particular cases of the following general fact:

**Proposition 1.5**

*If  $V(q, t) = V(x; q, t)$  is an operator on  $\Lambda$  with the property*

$$V(q, t) = \widetilde{V(0, t)}^q$$

*then*

$$\frac{V(q, t)^X \Omega[XY \frac{1-t}{1-q}]}{\Omega[XY \frac{1-t}{1-q}]} = \frac{V(0, t)^X \Omega[XY(1-t)]}{\Omega[XY(1-t)]}$$

*in particular this ratio is independent of  $q$ .*

The proof follows exactly the same steps we used to prove Proposition 1.1.

## 2. Applications

It is shown in [12] ((4.14) p. 324) that

$$P_\mu[X; t, t] = S_\mu[X] . \quad 2.1$$

thus 1.5 gives

$$J_\mu[X; t, t] = \prod_{s \in \mu} (1 - t^{h_\mu(s)}) S_\mu[X] , \quad 2.2$$

where  $h_\mu(s) = a_\mu(s) + l_\mu(s) + 1$  denotes the hook-length corresponding to the cell  $s$  in  $\mu$ . This given, Theorem 1.1 has the following immediate corollary.

### Theorem 2.1

*If  $B_k$  is any operator on  $\Lambda$  with the property*

$$B_k(X; t) J_\mu[X; 0, t] = J_{\mu+1^k}[X; 0, t] \quad \text{for all } l(\mu) \leq k \quad 2.3$$

*then the operator  $\widetilde{B}_k^t$  defined by setting for  $P \in \Lambda$*

$$\widetilde{B}_k^t(X; t) P[X] = B_k(Y; t)^Y P[tX + (1-t)Y] \Big|_{Y=X} \quad 2.4$$

*has the property*

$$\widetilde{B}_k^t(X; t) S_\mu[X] = \left( \prod_{i=1}^k (1 - t^{k+1-i+\mu_i}) \right) S_{\mu+1^k}[X] \quad \text{for all } l(\lambda) \leq k . \quad 2.5$$

### Proof

This follows by setting  $q = t$  in 1.9, using formula 2.2 and canceling the common factor.

This result has the following converse

### Theorem 2.2

*If  $B_k(X; t)$  is any operator with the property in 2.5, then the operator  $\widetilde{B}_k^q(X, t)$  defined by setting for  $P \in \Lambda$ ,*

$$\widetilde{B}_k^q(X; t) P[X] = B_k(Y; t)^Y P[qX + (1-q)Y] \Big|_{Y=X} \quad 2.6$$

*has the property*

$$\widetilde{B}_k^q(X; t) J_\mu[X; q, t] = J_{\mu+1^k}[X; q, t] \quad \text{for all } l(\mu) \leq k \quad 2.7$$

### Proof

The argument follows almost verbatim what we did to prove Theorem 1.1 except that at one point we must set  $q = t$  rather than  $q = 0$ .

Of course we may produce a polynomiality proof based on this theorem, however all our attempts yielded a more complicated argument than that based on Theorem I.1.

For representation theoretical reasons Garsia-Haiman where led to consider the polynomials

$$\tilde{H}_\mu[X; q, t] = \sum_{\lambda} S_\lambda[X] \tilde{K}_{\lambda\mu}(q, t) \quad 2.8$$

where

$$\tilde{K}_{\lambda\mu}(q, t) = t^{n(\mu)} K_{\lambda\mu}(q, 1/t) \quad 2.9$$

with

$$n(\mu) = \sum_i (i-1)\mu_i \quad 2.10$$

Note that setting  $q = 0$  in 2.8 gives

$$\tilde{H}_\mu[X; 0, t] = \tilde{H}_\mu[X; t] = \sum_{\lambda} S_\lambda[X] \tilde{K}_{\lambda\mu}(t)$$

where

$$\tilde{K}_{\lambda\mu}(t) = t^{n(\mu)} K_{\lambda\mu}(1/t)$$

is the so-called *cocharge* Kostka-Foulkes polynomial. Now it develops that a result analogous to Theorem I.1 holds also for the basis  $\{\tilde{H}_\mu[X; q, t]\}_\mu$ .

### Theorem 2.3

*If  $\mathcal{H}_k = \mathcal{H}_k(X, t)$  is any linear operator on  $\Lambda$  with the property that*

$$\mathcal{H}_k \tilde{H}_\mu[X; t] = \tilde{H}_{\mu+1^k}[X; t] \quad \text{for all } \mu \text{ of length } \leq k \quad 2.11$$

*then the operator  $\widetilde{\mathcal{H}}_k^q$  defined by I.11 has the property*

$$\widetilde{\mathcal{H}}_k^q \tilde{H}_\mu[X; q, t] = \tilde{H}_{\mu+1^k}[X; q, t] \quad \text{for all } \mu \text{ of length } \leq k. \quad 2.12$$

*In particular, the modified Macdonald polynomial  $\tilde{H}_\mu[X; q, t]$  may be obtained from the “Rodriguez” formula:*

$$\tilde{H}_\mu[X; q, t] = \widetilde{\mathcal{H}}_{\mu'_1}^q \widetilde{\mathcal{H}}_{\mu'_2}^q \cdots \widetilde{\mathcal{H}}_{\mu'_h}^q \mathbf{1} \quad 2.13$$

*where  $\mu' = (\mu'_1, \mu'_2, \dots, \mu'_h)$  denotes the conjugate of  $\mu$ .*

### Proof

Note that setting  $t = 1/t$  in 2.11 gives

$$t^{-n(\mu)} \mathcal{H}_k(X; 1/t) H_\mu[X; t] = t^{-n(\mu+1^k)} H_{\mu+1^k}[X; t]. \quad \text{for all } \mu \text{ of length } \leq k$$

and this (using 2.10) may be rewritten as

$$t^{\binom{k}{2}} \mathcal{H}_k(X; 1/t) H_\mu[X; t] = H_{\mu+1^k}[X; t] \quad \text{for all } \mu \text{ of length } \leq k.$$

Thus we may apply Theorem I.1 and derive that

$$t^{\binom{k}{2}} \mathcal{H}_k(Y; 1/t)^Y H_\mu[qX + (1-q)Y; q, t] \Big|_{Y=X} = H_{\mu+1^k}[X; q, t] \quad \text{for all } \mu \text{ of length } \leq k.$$

Setting  $t = 1/t$  and using 2.9 transforms this back to 2.12.

Some interesting developments follow by combining the present identities with one of the simplest of the “Rodriguez” operators introduced by Lapointe-Vinet in [10]. To see how this comes about we need to recall this beautiful result.

**Theorem 2.4**

*The operator*

$$LV_k(X_n; q, t) = \frac{1}{(\frac{1}{q}, \frac{1}{t})_{n-k}} M^{(X_n)} \left( -\frac{1}{qt^{n-k-1}} \right) \underline{e_k[X_n]} \quad 2.14$$

where  $\underline{e_k[X_n]}$  denotes multiplication by  $e_k[X_n]$ , has the property

$$LV_k(X_n; q, t) J_\mu[X_n; q, t] = J_{\mu+1^k}[X_n; q, t] \quad \text{for all } l(\mu) \leq k$$

**Proof**

The argument is so elementary that it might as well be reproduced here. It is shown by Macdonald in [12] (p. 340 (6.24) (iv)) that

$$e_k[X_n] P_\mu[X_n; q, t] = P_{\mu+1^k}[X_n; q, t] + \sum_{\substack{\lambda/\mu \in V_k \\ \lambda \neq \mu+1^k}} P_\lambda[X_n; q, t] \Psi_{\lambda\mu}(q, t)$$

where  $\lambda/\mu \in V_k$  indicates that the sum is over partitions  $\lambda$  such that  $\lambda/\mu$  is a vertical  $k$ -strip. Now applying the Macdonald operator  $M^{(X_n)}(u)$  to both sides and using (4.15) p. 324 of [12] we get for  $l(\mu) \leq k$

$$\begin{aligned} M^{(X_n)}(u) e_k[X_n] P_\mu[X_n; q, t] &= \prod_{i=1}^k (1 + u t^{n-i} q^{\mu_i+1}) \prod_{i=k+1}^n (1 + u t^{n-i}) P_{\mu+1^k}[X_n; q, t] \\ &\quad + \sum_{\substack{\lambda/\mu \in V_k \\ \lambda \neq \mu+1^k}} \prod_{i=1}^n (1 + u t^{n-i} q^{\lambda_i}) P_\lambda[X_n; q, t] \Psi_{\lambda\mu}(q, t). \end{aligned} \quad 2.15$$

Now  $1^k$  is the shortest vertical  $k$ -strip that may be added to  $\mu$ , and any other will spill a cell at height  $k+1$ . Thus each term of the sum in 2.15 will contain the factor

$$(1 + u t^{n-k-1} q)$$

which vanishes when we set  $u = -1/q t^{n-k-1}$ . Thus 2.15 gives

$$\frac{1}{\left(\frac{1}{q}, \frac{1}{t}\right)_{n-k}} M^{(X_n)}\left(-\frac{1}{q t^{n-k-1}}\right) e_k[X_n] P_\mu[X_n; q, t] = \prod_{i=1}^k (1 - t^{k+1-i} q^{\mu_i}) P_{\mu+1^k}[X_n; q, t]$$

and this is easily converted to 2.14 by means of I.5.

To state and prove our next result we need some notation. To begin, if  $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$  is a permutation in the symmetric group  $\mathcal{S}_n$  and  $a = (a_1, a_2, \dots, a_n)$  is a given vector, we set

$$\sigma a = (a_{\sigma_1}, a_{\sigma_2}, \dots, a_{\sigma_n}),$$

we also let  $Supp(a)$  denote the “support” of  $a$ , that is the set of elements

$$Supp(a) = \{i : a_i \neq 0\}.$$

Next, for any two vectors  $a = (a_1, a_2, \dots, a_n)$  and  $b = (b_1, b_2, \dots, b_n)$  we shall write  $a \approx b$  if and only if the components of  $b$  are a rearrangement of the components of  $a$ . More precisely, we set  $a \approx b$  if and only if for some  $\sigma \in \mathcal{S}_n$  we have

$$b = \sigma a$$

With this notation the monomial symmetric function  $m_\lambda$  may be represented by the sum

$$m_\lambda[X_n] = \sum_{p \approx \lambda} x^p. \quad 2.16$$

Finally, we shall generically denote by  $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)$  the indicator vector of a subset of  $\{1, 2, \dots, n\}$ . Thus setting  $|\epsilon| = k$  will mean that  $\epsilon$  represents a subset of cardinality  $k$ . In particular, the elementary symmetric function  $e_k$  may be represented by the sum

$$e_k[X_n] = \sum_{|\epsilon|=k} x^\epsilon \quad 2.17$$

It develops that, notwithstanding the presence of terms  $1/q$  in 2.14, we can evaluate the limit of the operator  $LV_k(X_n; q, t)$  as  $q \rightarrow 0$ . What follows is the following surprising corollary of the Lapointe-Vinet result.

**Theorem 2.5**

*Let  $TLV_k(t)$  be the operator defined by setting for the monomial basis*

$$TLV_k(t) m_\lambda = 0 \quad \text{when } l(\lambda) > k \quad 2.18$$

and

$$TLV_k(t) m_\lambda = \sum_{|\epsilon|=k} \sum_{p \approx \lambda} c_{\epsilon, p}(t) S_{p+\epsilon} \quad \text{when } l(\lambda) \leq k \quad 2.19$$

where  $S_{p+\epsilon}$  denotes the corresponding signed Schur function and

$$c_{\epsilon,p}(t) = \begin{cases} (-1)^{n-k} t^{\binom{n-k}{2}} \prod_{\epsilon_i=0} (-t^{k+1-i}) \prod_{\epsilon_i+p_i=1} (1-t^{k+1-i}) & \text{if } \text{Supp}(p) \subseteq \text{Supp}(\epsilon) \\ 0 & \text{otherwise} \end{cases} \quad 2.20$$

then we also have

$$TLV_k Q_\mu[X; t] = \begin{cases} Q_{\mu+1^k}[X; t] & \text{if } l(\mu) \leq k \\ 0 & \text{otherwise} \end{cases} \quad 2.21$$

### Proof

The original definition, of the Macdonald operator, given in (3.2) p. 315 of [12] may be written in the form

$$M^{(X_n)}(u) = \frac{1}{\Delta_n(x)} \sum_{\sigma \in S_n} \text{sign}(\sigma) x^{\sigma\delta} \prod_{i=1}^n (1 + ut^{n-\sigma_i} T_i^q) \quad 2.22$$

where  $\Delta_n(x)$  denotes the Vandermonde determinant in the variables  $x_1, \dots, x_n$ ,  $T_i^q$  denotes the operator that replaces  $x_i$  by  $qx_i$  and for convenience we have set

$$\delta = (n-1, n-2, \dots, 1, 0).$$

This given, taking account of 2.16 and 2.17 we may write the action of the Lapointe-Vinet operator  $LV_k$  on the monomial basis in the form

$$LV_k m_\lambda[X_n] = \frac{1}{(\frac{1}{q}, \frac{1}{t})_{n-k}} \sum_{|\epsilon|=k} \sum_{p \approx \lambda} \frac{1}{\Delta_n(x)} \sum_{\sigma \in S_n} \text{sign}(\sigma) x^{\sigma\delta} \prod_{i=1}^n (1 - t^{k+1-\sigma_i} q^{p_{\sigma_i} + \epsilon_{\sigma_i} - 1}) x^{\sigma(p+\epsilon)}. \quad 2.23$$

Since we may set

$$\frac{1}{\Delta_n(x)} \sum_{\sigma \in S_n} \text{sign}(\sigma) x^{\sigma(\delta+p+\epsilon)} = S_{p+\epsilon}[X_n]$$

we see that 2.23 reduces to

$$LV_k(q, t) m_\lambda[X_n] = \sum_{|\epsilon|=k} \sum_{p \approx \lambda} c_{\epsilon,p}(q, t) S_{p+\epsilon}[X_n] \quad 2.24$$

with

$$c_{\epsilon,p}(q, t) = \frac{1}{(\frac{1}{q}, \frac{1}{t})_{n-k}} \prod_{i=1}^n (1 - t^{k+1-i} q^{p_i + \epsilon_i - 1}). \quad 2.25$$

Our next step is to evaluate  $c_{\epsilon,p}(q, t)$  at  $q = 0$ . To this end it is convenient to rewrite 2.25 in the form

$$c_{\epsilon,p}(q, t) = \frac{(-1)^{n-k} t^{\binom{n-k}{2}}}{(q; t)_{n-k}} \frac{q^{n-k}}{q^{\#\{i: \epsilon_i + p_i = 0\}}} \prod_{\epsilon_i + p_i > 1} (1 - t^{k+1-i} q^{p_i + \epsilon_i - 1}) \times \prod_{\epsilon_i + p_i = 1} (1 - t^{k+1-i}) \prod_{\epsilon_i + p_i = 0} (q - t^{k+1-i}) \quad 2.26$$

Now note that since  $|\epsilon| = k$  we necessarily have that

$$\#\{i : \epsilon_i + p_i = 0\} \leq n - k$$

with equality only if

$$Supp(p) \subseteq Supp(\epsilon) . \tag{2.27}$$

Thus it follows that

$$\frac{q^{n-k}}{q^{\#\{i : \epsilon_i + p_i = 0\}}} \Big|_{q=0} = 0$$

when 2.27 fails to hold. Since we have the obvious evaluations

$$\begin{aligned} (q; t)_{n-k} \Big|_{q=0} &= 1 \\ \prod_{\epsilon_i + p_i > 1} (1 - t^{k+1-i} q^{p_i + \epsilon_i - 1}) \Big|_{q=0} &= 1 \\ \prod_{\epsilon_i + p_i = 0} (q - t^{k+1-i}) \Big|_{q=0} &= \prod_{\epsilon_i + p_i = 0} (-t^{k+1-i}) \end{aligned}$$

we see that from 2.26 we derive that

$$c_{\epsilon, p}(0, t) = c_{\epsilon, p}(t) ,$$

with  $c_{\epsilon, p}(t)$  precisely as defined in 2.20. Thus from 2.24 we get

$$LV_k(0, t) m_\lambda[X_n] = \sum_{|\epsilon|=k} \sum_{p \approx \lambda} c_{\epsilon, p}(0, t) S_{p+\epsilon}[X_n] = TLV_k(t) m_\lambda[X_n] .$$

To show 2.21 note that from I.8 we get

$$LV_k(q, t) J_\mu[X_n; q, t] = \sum_{\lambda} \left( LV_k(q, t) S_\lambda[X_n(1-t)] \right) K_{\lambda\mu}(q, t) .$$

Thus the polynomiality of the  $K_{\lambda\mu}(q, t)$  assure that we can safely set  $q = 0$  here and obtain that

$$\begin{aligned} LV_k(q, t) J_\mu[X_n; q, t] \Big|_{q=0} &= \sum_{\lambda} \left( LV_k(0, t) S_\lambda[X_n(1-t)] \right) K_{\lambda\mu}(0, t) \\ &= \sum_{\lambda} \left( TLV_k(t) S_\lambda[X_n(1-t)] \right) K_{\lambda\mu}(t) \\ &= TLV_k(t) Q_\mu[X_n; t] \end{aligned}$$

Thus when  $l(\mu) \leq k$  the Lapointe-Vinet result (Theorem 2.5) yields

$$TLV_k(t) Q_\mu[X; t] = J_{\mu+1^k}[X; 0, t] = Q_{\mu+1^k}[X; t] .$$



This proves the first alternative in 2.21. To show the second we note that (2.6) of [12] p. 209 implies that we have an expansion of the form

$$J_\mu[X; t] = \sum_{\lambda \leq \mu} m_\lambda[X] \xi_{\lambda\mu}(t) . \quad 2.28$$

Thus when  $l(\mu) > k$  we shall have  $l(\lambda) > k$  for all the summands in 2.28 and 2.18 then gives that

$$TLV_k(t) J_\mu[X_n; t] = 0 .$$

This completes our proof.

We should note that the Lapointe-Vinet result has one further curious consequence.

**Theorem 2.6**

Let  $W_k = W_k(X_n; t)$  be the operator defined by setting for every  $P \in \Lambda$

$$W_k P[X] = \frac{1}{(\frac{1}{t}; \frac{1}{t})_{n-k}} \sum_{r=0}^n \left( \frac{-1}{t^{n-k}} \right)^r \sum_{\substack{I \subseteq [1, n] \\ |I|=r}} B_I(x; t) P[X - X_I] \quad 2.29$$

with

$$B_I(x; t) = \frac{1}{\Delta_n(x)} T_I^t \Delta_n(x) e_k[X_n] . \quad 2.30$$

Then

$$\widetilde{W}_k^q J_\mu[X_n; q, t] = J_{\mu+1^k}[X_n; q, t] \quad \text{for all } \mu \text{ of length } \leq k. \quad 2.31$$

**Proof**

Using formula (3.5) of [12] p. 316 we may write the Lapointe-Vinet result (for  $l(\mu) \leq k$ ) in the form

$$J_{\mu+1^k}[X_n; q, t] = \frac{1}{(\frac{1}{q}; \frac{1}{t})_{n-k}} \sum_{r=0}^n \left( \frac{-1}{q t^{n-k-1}} \right)^r \sum_{\substack{I \subseteq [1, n] \\ |I|=r}} \frac{T_I^t \Delta_n(x)}{\Delta_n(x)} T_I^q e_k[X_n] J_\mu[X_n; q, t]$$

where, for each  $i \in I$ ,  $T_I^t$  replaces  $x_i$  by  $tx_i$ . This given, we may set  $q = t$  and obtain that

$$\begin{aligned} J_{\mu+1^k}[X_n; t, t] &= \frac{1}{(\frac{1}{t}; \frac{1}{t})_{n-k}} \sum_{r=0}^n \left( \frac{-1}{t^{n-k}} \right)^r \sum_{\substack{I \subseteq [1, n] \\ |I|=r}} B_I(x; t) J_\mu[X_n + (t-1)X_I; t, t] \\ &= \widetilde{W}_k^t J_\mu[X_n; t, t] . \end{aligned} \quad 2.32$$

Since

$$J_\mu[X_n; t, t] = h_\mu(t) S_\mu[X_n] ,$$

we derive from 2.32 that the operator  $W_k$  satisfies the hypothesis of Theorem 2.2, thus 2.31 is an immediate consequence of Theorem 2.2.

We terminate with one final application of Theorem I.1:

**Theorem 2.7**

*Let  $G_k(X; t)$  be any operator with the property that*

$$G_k(X; t)H_\mu[X; t] = H_{\mu+1^k}[X; t] \quad \text{for all } \mu \text{ of length } \leq k. \quad 2.33$$

*Then the operator*

$$\widetilde{G}_k^t(X; q) = \widetilde{G}_k^q(X; t) \Big|_{t \rightleftharpoons q}$$

*has the property*

$$\omega \widetilde{G}_k^t(X; q) \omega H_\mu[X; q, t] = H_{(k, \mu)}[X; q, t] \quad \text{for all } \mu \text{ with } \mu_1 \leq k. \quad 2.34$$

**Proof**

It follows from the Macdonald duality formula [12] ((5.1) p. 327) that the polynomial  $H_\mu[X; q, t]$  satisfies the identity

$$H_{\mu'}[X; q, t] = \omega H_\mu[X; t, q] \quad 2.35$$

Now, assuming 2.33, from Theorem I.1 it follows that

$$\widetilde{G}_k^q(X; t) H_{\mu'}[X; q, t] = H_{\mu'+1^k}[X; q, t] \quad \text{for all } \mu' \text{ of length } \leq k.$$

Interchanging  $q$  and  $t$  we get

$$\widetilde{G}_k^t(X; q) H_{\mu'}[X; t, q] = H_{\mu'+1^k}[X; t, q] \quad \text{for all } \mu' \text{ of length } \leq k.$$

and two uses of 2.35 then give

$$\widetilde{G}_k^t(X; q) \omega H_\mu[X; t, q] = \omega H_{(k, \mu)}[X; q, t] \quad \text{for all } \mu \text{ with } \mu_1 \leq k,$$

which simply another way of writing formula 2.34.

The simplicity of our operator  $\omega \widetilde{G}_k^t(X; q) \omega$  and our proof of 2.34 should be contrasted with the complexity of the developments in the recent Kajihara-Noumi paper [5].

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